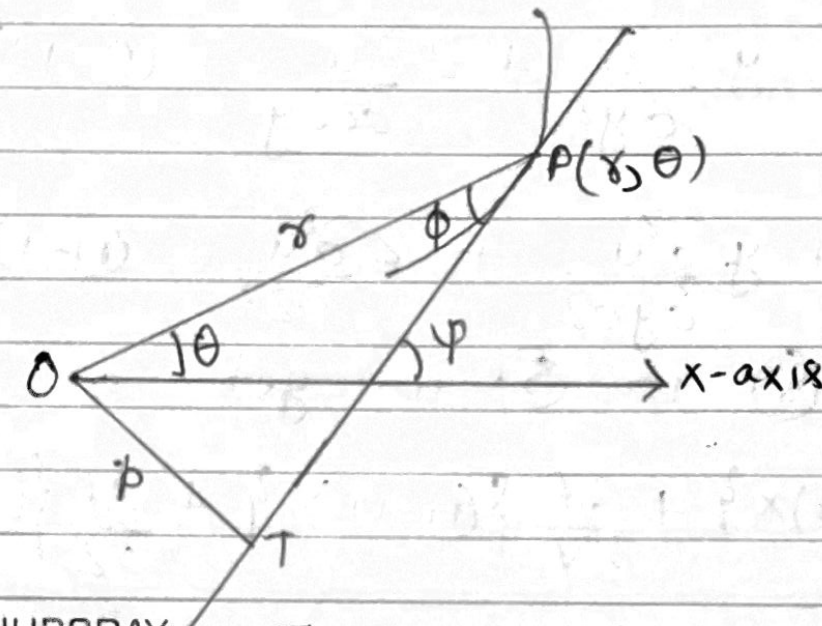
Zoom Class 22/04/2020

Pedal Formula for radius of Curvature



The above figure represents the curve with pedal eqn $p = f(r)$.
Let $P(r, \theta)$ be any point on the curve and $\angle POX = \theta$. Draw a tangent PT at an angle ψ with x-axis and ϕ with OP.

then from the figure we get $\psi = \phi + \theta$

$$\therefore \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \quad \text{—— (1)}$$

Draw $OT \perp PT$ where $OT = p$

$$\Rightarrow p = r \sin\phi \quad \text{—— (2)}$$

Differentiating (2) w.r.t. 'x' we get

$$\frac{dp}{dr} = r\cos\phi \frac{d\phi}{dr} + \sin\phi$$

$$\Rightarrow \frac{dp}{dr} = r \cdot \frac{dr}{ds} \cdot \frac{d\phi}{dr} + r\frac{d\theta}{ds} \longrightarrow \left\{ \because \frac{dr}{ds} = \cos\phi \;\&\; r\frac{d\theta}{ds} = \sin\phi \right\}$$

$$= r\frac{d\phi}{ds} + r\frac{d\theta}{ds}$$

$$= r\left\{\frac{d\phi}{ds} + \frac{d\theta}{ds}\right\}$$

$$= r\frac{d\psi}{ds} \quad \text{— from eqn (1)}$$

$$\Rightarrow \frac{dp}{dr} = r \cdot \frac{1}{\rho}$$

$$\Rightarrow \rho = r \cdot \frac{dr}{dp}$$

which is the required pedal formula.

Polar Tangential formula for 'ρ'

If the given equation of curve is in terms of 'p' and 'ψ' then the equation is of polar tangential form.

Now, let the equation of curve be $p = f(\psi)$

$$\Rightarrow \frac{dp}{d\psi} = \frac{dp}{dr} \cdot \frac{dr}{ds} \cdot \frac{ds}{d\psi}$$

$$= \frac{dp}{dr} \cdot \cos\phi \cdot \rho \longrightarrow \left\{ \because \frac{dr}{ds} = \cos\phi \;\&\; \frac{ds}{d\psi} = \rho \right\}$$

$$= \frac{dp}{dr} \cos\phi \cdot r \cdot \frac{dr}{dp} \longrightarrow \left\{ \text{value of } \rho \text{ in pedal form} \right\}$$

$$\Rightarrow \frac{dp}{d\psi} = r\cos\phi \quad \text{--- (1)}$$

Now since $p = r\sin\phi$ — (2)

$$\Rightarrow p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi$$
$$= r^2\{\sin^2\phi + \cos^2\phi\}$$

$$\Rightarrow p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2 \quad \text{--- (3)}$$

Diffrentiating '(3)' w.r.t. 'p' we get

$$2p + \frac{d}{dp}\left\{\left(\frac{dp}{d\psi}\right)^2\right\} = 2r\frac{dr}{dp}$$

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$$\Rightarrow 2p + \frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)^2 \cdot \frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

$$\Rightarrow 2p + 2\cancel{\frac{dp}{d\psi}} \cdot \frac{d^2p}{d\psi^2} \cdot \cancel{\frac{d\psi}{dp}} = 2r\frac{dr}{dp}$$

$$\Rightarrow \cancel{2}\left\{p + \frac{d^2p}{d\psi^2}\right\} = \cancel{2}r\frac{dr}{dp}$$

$$\Rightarrow \boxed{p + \frac{d^2p}{d\psi^2} = \rho}$$

is the required eqn.

Polar formula for radius of Curvature

The equation of curve in $r = f(\theta)$ is said to be in polar form.

We know $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ —— ①

Diffrentiating ① w.r.t. θ we get

$$\Rightarrow \frac{-2}{p^3}\frac{dp}{d\theta} = \frac{-2}{r^3}\frac{dr}{d\theta} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^3 + \frac{2}{r^4}\left(\frac{dr}{d\theta}\right)\frac{d^2r}{d\theta^2}$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{d\theta} = \frac{1}{r^3}\left(\frac{dr}{d\theta}\right) + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^3 - \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)\left(\frac{d^2r}{d\theta^2}\right)$$

$$\Rightarrow \frac{1}{p^3}\cdot\frac{dp}{\cancel{d\theta}}\cdot\frac{\cancel{d\theta}}{dr}$$

$$= \frac{1}{r^3}\cancel{\left(\frac{dr}{d\theta}\right)}\cancel{\left(\frac{d\theta}{dr}\right)} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^{\cancel{3}\,2}\cdot\cancel{\left(\frac{d\theta}{dr}\right)} - \frac{1}{r^4}\cancel{\left(\frac{dr}{d\theta}\right)}\cancel{\left(\frac{d\theta}{dr}\right)}\left(\frac{d^2r}{d\theta^2}\right)$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{d^2r}{d\theta^2}$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^5}\left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)\right\}$$

$$\Rightarrow \frac{dp}{dr} = \frac{p^3}{r^5}\left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)\right\} \quad \text{—— ②}$$

Now ① $\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2 + (dr/d\theta)^2}{r^4}$

$\Rightarrow p^2 = \frac{r^4}{(r^2 + (dr/d\theta)^2)} \Rightarrow p = \frac{r^2}{\{r^2 + (dr/d\theta)^2\}^{1/2}}$ —③

Now ② $\Rightarrow \frac{dr}{dp} = \frac{r^5}{p^3} \cdot \frac{1}{\{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\}}$

$\Rightarrow r\frac{dr}{dp} = \frac{r^6}{p^3} \cdot \frac{1}{\{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\}}$

$\Rightarrow \rho = \frac{\cancel{r^6}}{\cancel{(r^2)^3}} \times \frac{[\{r^2 + (dr/d\theta)^2\}^{1/2}]^3}{\{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\}}$ — {using ③}

$\Rightarrow \rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{\{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\}}$

$\Rightarrow \boxed{\rho = \frac{\{r^2 + r_1^2\}^{3/2}}{r^2 + 2r_1^2 - r r_2}}$ where $r_1 = dr/d\theta$
$r_2 = d^2r/d\theta^2$

is the required eqn.